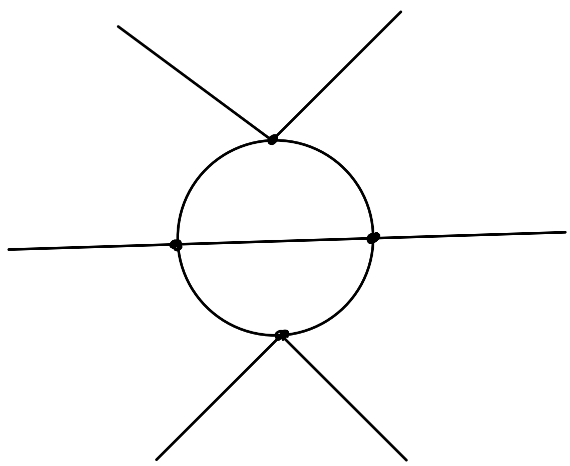
Theorem: $D = 4 - E$

where D: superficial degree of divergence
E: number of external lines

Proof:

Let us define I as the number of internal lines, V as number of vertices, L as number of loops.

example diagram:



$E = 6$
$L = 2$
$I = 5$
$V = 4$

$\longrightarrow$ number of loops L = number of integrals $\int \left[\frac{d^4k}{(2\pi)^4}\right]$

we have $L = I - (V - 1)$ overall mom. cons.

carries momentum to be integrated over

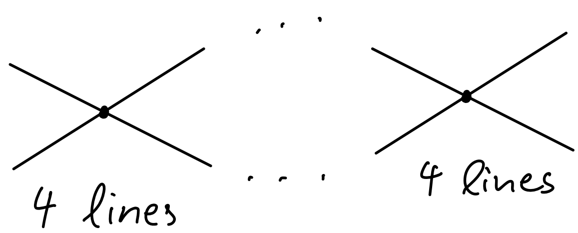
carries mom. cons. delta fkt.

For example, for the above diagram the eq. gives:

$$\underset{\substack{\| \\ 2}}{L} = \underset{\substack{\| \\ 5}}{I} - \underset{\substack{\| \\ 4}}{(V} - 1) \qquad (1)$$

so $\quad 2 = 5 - (4-1) \quad \checkmark$

→ for each vertex there are 4 lines coming in (or going out)



since internal lines connect vertices, they count twice, giving

$$4V = E + 2I \qquad (2)$$

(our diagram: $4.4 = 6 + 2 \cdot 5$)

→ finally, for each loop there is a $\int d^4k$ while for each internal line there is $i/(k^2 - m^2 + i\varepsilon)$, giving

$$D = 4L - 2I \qquad (3)$$

Putting (1), (2), (3) together gives:

$$D = 4(I - (V-1)) - 2I = 4\left(I - \left(\tfrac{1}{4}E + \tfrac{1}{2}I\right) - 1\right) - 2I = 4 - E \qquad \square$$

Degree of divergence with fermions

Consider now the Yukawa theory with Lagr.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_p)\psi + \frac{1}{2}\left[(\partial\varphi)^2 - \mu_p^2\varphi^2\right] - \lambda_p\varphi^4 + f_p\varphi\bar{\psi}\psi \qquad (4)$$

→ we have to count

E_f : number of external fermion lines

I_f : number of internal fermion lines

V_f : vertices with coupling f

V_λ : vertices with coupling λ

→ eq. (1) becomes

$$L = I_f + I_b - (V_f + V_\lambda - 1)$$

→ eq. (2) becomes

$$V_f + 4V_\lambda = E_b + 2I_b$$

and

$$2V_f = E_f + 2I_f$$

→ eq. (3) becomes

$$D = 4L - I_f - 2I_b$$

$$\rightarrow D = 4L - I_f - 2I_b$$
$$= 4\big((I_f + I_b) - (V_f + V_\lambda - 1)\big) - I_f - 2I_b$$
$$= 4\left[I_f + I_b - \left(\tfrac{1}{2}E_f + I_f\right) - \tfrac{1}{4}\left(E_b - \tfrac{1}{2}E_f + 2I_b - I_f\right) + 1\right] - I_f - 2I_b$$
$$= 4I_f + 4I_b - 2E_f - 4I_f - E_b + \tfrac{1}{2}E_f - 2I_b + I_f + 4 - I_f - 2I_b$$
$$= 4 - \tfrac{3}{2}E_f - E_b$$

So the divergent diagrams with $D \geq 0$ are those for which

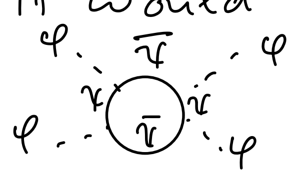
$$(E_b, E_f) = (0,2),\ (2,0),\ (1,2) \text{ and } (4,0)$$

$\rightarrow$ these correspond to the 6 terms in the Lagrangian (4) !

$\rightarrow$ we need 6 counterterms

Suppose we hadn't included the $\lambda_b \varphi^4$ interaction

$\rightarrow$ it would have been generated by



$\rightarrow$ have to include $\lambda_b \varphi^4$ as counterterm !

Nonrenormalizable field theories

Consider the Fermi theory of the weak interaction:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_p)\psi + G(\bar{\psi}\psi)^2$$

$\longrightarrow$ the analogs of eqs. (1), (2), (3) now read:

$$L = I_f - (V-1)$$
$$4V = E_f + 2I_f$$
$$D = 4L - I_f$$

giving

$$D = 4 - \frac{3}{2}E_f + 2V$$

we see that D now depends on V

$\longrightarrow$ amplitude for fermi-fermi scattering ($E_f = 4$) gets worse and worse as we go to higher order in perturbation series

$\longrightarrow$ for any E_f we get divergent diagrams if V is sufficiently large!

$\longrightarrow$ have to include infinitely many counterterms $(\bar{\psi}\psi)^3$, $(\bar{\psi}\psi)^4$, $(\bar{\psi}\psi)^5$, ...

Dependence on dimension

the superficial degree of divergence depends on dimension d of space time

$\longrightarrow \int d^d k$ for each loop is d-dep.

Consider, for example, Fermi interaction $G(\bar{\psi}\psi)^2$ in $d=(1+1)$

$$\longrightarrow \quad L = I_f - (V-1)$$

$$4V = E_f + 2I_f$$

$$D = 2L - I_f$$

$$= 2\left(I_f - (V-1)\right) - I_f$$

$$= 2\left(I_f - \tfrac{1}{4}\left(E_f + 2I_f - 4\right)\right) - I_f$$

$$= \cancel{2I_f} - \tfrac{1}{2}E_f - \cancel{I_f} + 2 - \cancel{I_f}$$

$$= 2 - \tfrac{1}{2}E_f$$

$\longrightarrow$ no longer depends on V!
Fermi interaction is renormalizable in $d=(1+1)$!

General analysis (for bosons)

Consider a theory with pure φ^r interaction in d dimensions

$$\longrightarrow \quad I = \frac{1}{2}(Vr - E)$$

$$L = I - (V-1)$$

$$D = Ld - 2I$$

and hence

$$D = (I - (V-1))d - 2I$$

$$= \left(\frac{1}{2}(Vr - E) - (V-1)\right)d - 2\frac{1}{2}(Vr - E)$$

$$= V\left(\frac{1}{2}rd - r - d\right) + \left(d + E - \frac{1}{2}Ed\right)$$

$$= -V\underbrace{\left(d + r - \frac{1}{2}rd\right)}_{=[\lambda]} + \left(d + E - \frac{1}{2}Ed\right)$$

$$[\partial_\mu \varphi \partial^\mu \varphi] = d \longrightarrow [\varphi] = \frac{1}{2}d - 1$$

$$[\lambda \varphi^r] = d \longrightarrow [\lambda] = d - \frac{1}{2}rd + r$$

$$\overset{!}{=} -V[\lambda] + \left(d + E - \frac{1}{2}Ed\right)$$

$\longrightarrow$ iff $[\lambda] = 0$, D is independent of V, this happens for $d_c = \frac{2r}{r-2}$ or $d_c = 4$ for φ^4 theory, $d_c = 6$ for φ^3 theory, and $d_c = 3$ for φ^6 theory, ...

$\longrightarrow$ at $d = d_c$ only finitely many counterterms needed $\to$ "renormalizable"

- at $d > d_c$ "non-renormalizable"
- at $d < d_c$ "super-renormalizable"